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MODE-III CRACK KINKING WITH DELAY TIME: AN ANALYTICAL APPROXIMATION

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Abstract—The singular part of the elastodynamic field in the vicinity of a propagating crack tip plays an important role in fracture mechanics considerations. The dynamic solution near a crack tip which branches or kinks is required to understand the observed bifurcation events in brittle materials. We consider a rather general time dependent stress wave loading incident at an arbitrary angle on a semi-infinite crack in a linear elastic solid. To model the kinking of a stationary crack under stress wave loading correctly, a delay time for initiation of the new crack must be included which means the problem loses the property of self-similarity and makes it significantly more difficult. A perturbation method is used to obtain the dynamic stress intensity factor for the kinking crack. The method relies on solving simple problems which can be used with linear superposition to solve the problem of a kinked crack. The solution is represented in a simple closed form as a function of the incident angle α of the stress wave, the kink angle δ, the kinking crack speed v_c and the finite delay time t_f. This gives more information about the effect of parameters on the solution than the purely numerical results. Finally, the maximum of the energy flux into the propagating kinked crack tip is found as a function of kink angle and crack tip velocity, and some implications of this are discussed.

INTRODUCTION

When a crack forms in a brittle material, such as glass and some plastics and metals at low temperatures, there appears (from experimental results, e.g. Ravi-Chandar and Knauss[1]) to be a maximum crack tip velocity which is significantly less than the theoretically expected maximum of the Rayleigh wave speed[2]. This maximum crack tip speed seems to prevent the crack tip from absorbing enough energy under high loads and the single crack becomes unstable. In an attempt to absorb more energy, it searches for alternative paths, and the observed behavior is the commonly seen bifurcation event[3, 4]. This phenomenon becomes particularly frequent when the speed of crack propagation becomes relatively large.

Another case where kinking occurs is when a stress wave of sufficiently high magnitude impacts on a stationary crack. Ravi-Chandar and Knauss[1, 5–8] have recently performed some carefully controlled experiments with dynamic loading of a long crack. Examples of such situations are earthquake generated stress waves on faults in the earth's crust and impact loading of a body in which the loading is rapid enough to cause stress wave motion.

It has been shown (see [9], for example) that stress wave loading of a crack in a linear elastic material causes a dynamic overshoot of the stress intensity factor when compared with a similar loading applied quasi-statically. It is therefore possible that a crack which will remain stationary under quasistatic loading, may become unstable under dynamic loading and propagate. The direction of propagation, as well as the velocity of crack propagation, at the instant of initiation will depend on the local stress field around the crack tip. Experimental observations of the magnitude of the speed of crack propagation at branching suggest that elastodynamic effects play a significant role. It has been observed[3, 10, 11] that the angle of branches of the bifurcated crack is approximately $25° \sim 40°$, and the velocity of the bifurcated crack tips is about 10% less than the crack tip velocity just prior to bifurcation.

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The problem of a crack that is branching at an arbitrary angle is difficult to solve; hence the analytical work is rather limited, and most of it is elastostatic in nature. For antiplane strain, elastostatic analyses were presented by Sih[12] and Smith[13]. The in-plane problems were treated by Cotterell and Rice[14] who used a perturbation technique, similar to that proposed independently by Banichuk[15] to derive the conditions necessary for slightly out-of-plane quasistatic growth of a semi-infinite straight crack under mixed-mode loading conditions. Karihaloo *et al.*[16] have extended these results to higher-order terms of the asymptotic expansion.

A great deal of progress has been made recently in analyzing the problem of a semi-infinite crack under stress wave loading. All the analyses require that the dynamic problem considered be self-similar in the variables r (distance from original crack tip) and time t. This puts a restriction on the problems that can be considered. For antiplane strain deformation, it has been considered by Burgers and Dempsey[17] analytically for bifurcation half-angles $= 0$, $\pi/2$ and numerically by Burgers for all the angles in between[18]. Subsequently Dempsey *et al.*[19] have used a conformal mapping to obtain the analytical solution for kinking crack under stress wave loading. More recently, the problem of asymmetric crack bifurcation under stress wave loading has been studied by Dempsey *et al.*[20]. The plane strain deformation of kinking case has been solved numerically by Burgers[21] and the bifurcation case by Burgers and Dempsey[22]. The analytical solution for this mixed Mode-I–II problem has not been found yet.

The transient elastodynamic nonplanar crack growth solutions mentioned above are all for the case when the geometry and loading conditions make the problem self-similar. That is, the loading must cause stresses or rates of stresses which are functions of θ and r/t only. For this to be so, the geometry must have no characteristic length associated with it. This property of self-similarity allows Chaplygin's transformation and a conformal mapping to be used to solve these problems analytically[17, 19, 20]. In all these models, it is assumed that the new crack initiates out of the original crack tip at an angle δ at the same time as loading is applied on the crack faces (or the stress wave loading is passing the original crack tip). It was observed by Achenbach[23] that if a plane stress pulse strikes the half-plane crack in an initially undisturbed medium, instantaneous crack propagation can occur only if the stress pulse front carries a square-root singular stress. Hence it will greatly improve the model by allowing a finite delay time in the initiation of the nonplanar crack. But if we do so, the problem loses its self-similar nature. The only solution for the newly initiated crack propagating after a delay time has been obtained by Freund[24], and this solution is restricted to the crack remaining straight.

We consider the dynamic crack growth out of the original semi-infinite crack at an angle to the original crack at some time after the dynamic loading is applied to the crack faces (or after the stress wave loading initially interacts with the crack tip). A perturbation method is used to obtain the first order analytic closed form solution near the kinking crack tip. By setting the finite delay time t_f to zero, this solution agrees closely with the semi-analytical numerical results in [25] where it is shown that the range of kinking angles and crack-tip speeds for which this approximation gives good results is surprisingly large in comparison with exact results[19]. The closed form solution gives more analytical information than the purely numerical results so that the effect of certain parameters on the solution can easily be seen. The elastodynamic stress intensity factors of the kinking crack tip are used to compute the corresponding fluxes of energy into the propagating crack tip. For a specific incident angle of the stress wave loading, the energy flux into the crack tip shows a distinct maximum at a particular combination of crack-kinking angle and crack-tip speed.

1. DEFINITION OF PROBLEM AND SOLUTION METHODOLOGY

Consider a stress-free linear elastic homogeneous isotropic infinite medium that contains a stationary semi-infinite crack, which will be referred to as the original crack. At time $t = 0$, an incident horizontally polarized transverse wave with angle α strikes the stationary crack tip. A short time later, at $t = t_f$, a new crack propagates out of the original semi-infinite crack. The velocity of propagation v_c is constant and less than the shear wave speed v_s. The line of propagation is straight, making an angle δ with the original crack, thus producing a kinked crack. The crack generates a plane reflected wave and a cylindrical diffracted wave. The pattern of wavefronts and the position of the crack tip for $t > t_f$ are shown in Fig. 1.

The propagation of horizontally polarized transverse waves in a homogeneous, isotropic, linearly elastic medium is governed by the two-dimensional wave equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}=b^2\frac{\partial^2 w}{\partial t^2}, \tag{1.1}$$

where b is the slowness of the transverse wave given by

$$b = 1/v_s = \sqrt{\rho/\mu}.$$

$w(x, z, t)$ is the displacement normal to the xz-plane; μ and ρ are the shear modulus and the mass density of the material, respectively. The nonvanishing stresses are

$$\tau_{xy} = \mu(\partial w/\partial x), \qquad \tau_{zy} = \mu(\partial w/\partial z). \tag{1.2}$$

The incident stress wave is of the form

$$\tau_{zy}^{\text{in}} = -\tau_0 H[t+(x/v_s)\sin\alpha-(z/v_s)\cos\alpha], \tag{1.3}$$

where $H(\,)$ is the Heaviside step function. We will extend the analysis to more general time dependence of the incident wave later. The method of solving this problem relies on an asymptotic approach. We use the perturbation procedure indicated by Kuo and Achenbach[26] which expresses the elastodynamic fields near a kinked crack in terms of a power series of δ. Each order in the approximation requires the field for a crack which propagates in its own plane, but where the crack faces are subjected to crack-face tractions which are related to the actual crack-kinking geometry. It is also shown that the second-order contribution to the Mode-III stress intensity factor vanishes, which corresponds to the first-order approximation of stress intensity factor accurate up to $o(\delta^2)$. Hence the

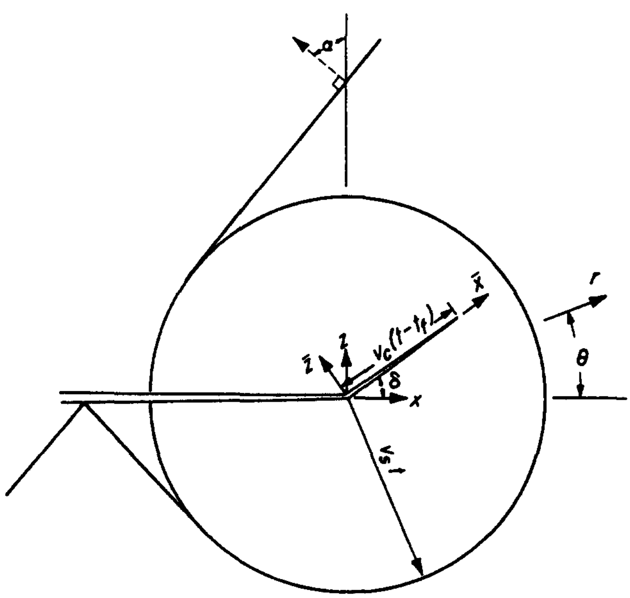

Fig. 1. Geometry of wavefronts for a kinking crack subject to an incident stress wave.

approximate results give good agreement with the exact solution up to large values of the kinking angle. This suggests that the elastodynamic stress intensity factor for kinked crack is affected more by dependence on the loading of the new crack faces than by the wedge geometry at the original crack tip.

The kinking of cracks under dynamic loading can be separated into a number of different problems all of which are relevant problems in their own right. By building up the so-called fundamental solutions of these more basic problems, the solution to the final complete problem can then be solved. The fundamental problems to be solved are the following:

(i) The full field solution for diffraction of a step-stress wave as given in (1.3) by the stationary semi-infinite crack. This problem can be analyzed by the use of integral transforms together with an application of the Wiener–Hopf technique and the Cagniard–de Hoop method of Laplace inversion. A detailed description of the approach can be found in [27, 28].

(ii) The solution for the concentrated forces acting on the crack tip at the instant that crack begins to extend at a constant speed. The forces have equal magnitudes, each being the same general linear function of time and move in the same direction as the crack propagating with a speed less than the crack speed. A similar problem has been solved in Mode-I by Freund[24].

The final step will be to combine the problems (i) and (ii) to solve for the stress intensity factor of the first-order approximation for the kinked crack that propagates with a constant velocity in an analytic closed form.

2. THE FUNDAMENTAL SOLUTIONS

For a stationary semi-infinite crack, the diffracted stress fields for the incident step-stress wave given in (1.3) are

$$\tau_{zy}^D(r,\theta,t) = D_1 \int_1^{t/br} \frac{-D_2 \cos\theta + \xi(\xi-1)\cos(\theta/2) + \xi \sin\alpha \cos(3\theta/2)}{(\xi-\sin\theta_1)(\xi+\sin\theta_2)(\xi-1)^{1/2}}\, d\xi, \tag{2.1}$$

$$\tau_{xy}^D(r,\theta,t) = -D_1 \int_1^{t/br} \frac{-D_2 \sin\theta + \xi(\xi-1)\sin(\theta/2) + \xi[\sin(\theta/2) + \sin\alpha \sin(3\theta/2)]}{(\xi-\sin\theta_1)(\xi+\sin\theta_2)(\xi-1)^{1/2}}\, d\xi, \tag{2.2}$$

where

$$D_1 = \frac{\tau_0 \cos\alpha}{\pi(1+\sin\alpha)^{1/2}}, \qquad D_2 = \sin\theta \sin(\theta/2) + \sin\alpha \cos(\theta/2),$$

$$\theta_1 = \theta - \alpha, \qquad \theta_2 = \theta + \alpha.$$

Combining τ_{zy}^D and τ_{xy}^D, we find

$$\begin{aligned}
\tau_{\theta y}^D &= \tau_{zy}^D \cos\theta - \tau_0^D \sin|\theta| \\
&= D_1 \int_1^{t/br} \frac{[\sin\theta \sin(\theta/2) + \sin\alpha \cos(\theta/2) + \xi \cos(\theta/2)](\xi-1)^{1/2}}{(\xi-\sin\theta_1)(\xi+\sin\theta_2)}\, d\xi \\
&= D_1 \left\{ 2\cos\frac{\theta}{2}\sqrt{\frac{t}{br}-1} - \frac{\sqrt{1-\sin\theta_1}}{\cos\alpha}\left[\sin\frac{\theta}{2} + \cos\left(\alpha - \frac{\theta}{2}\right)\right] \tan^{-1}\sqrt{\frac{t/br-1}{1-\sin\theta_0}} \right. \\
&\quad \left. - \frac{\sqrt{1+\sin\theta_2}}{\cos\alpha}\left[-\sin\frac{\theta}{2} + \cos\left(\alpha+\frac{\theta}{2}\right)\right]\tan^{-1}\sqrt{\frac{t/br-1}{1+\sin\theta_2}} \right\}.
\end{aligned} \tag{2.3}$$

As $r/bt \to 0$, (2.3) reduces to

$$\tau_{\theta y}^{D} = D_1\left\{2\cos\frac{\theta}{2}\sqrt{\frac{t}{br}} - \frac{\pi\sqrt{1-\sin\theta_1}}{2\cos\alpha}\left[\sin\frac{\theta}{2}+\cos\left(\alpha-\frac{\theta}{2}\right)\right]\right.$$
$$\left. - \frac{\pi\sqrt{1+\sin\theta_2}}{2\cos\alpha}\left[-\sin\frac{\theta}{2}+\cos\left(\alpha+\frac{\theta}{2}\right)\right]\right\}+o(1). \quad (2.4)$$

Denoting the stress intensity factor at time $t < t_f$ for stationary crack by $K^S(t)$, it is found that

$$\lim_{x\to 0^+}[\sqrt{2\pi x}\,\tau_{zy}^{D}(x,0,t)] \equiv K^S(t) = \frac{2\sqrt{2}\tau_0\cos\alpha t^{1/2}}{\sqrt{\pi(1+\sin\alpha)b}}. \quad (2.5)$$

Thus the stress intensity factor increases from zero as the square root of the time measured from the instant the wave strikes the crack. The main reason for obtaining the second and third term [which is of $O(1)$] in the asymptotic expansion (2.4) is that these two terms will later play a significant role in the calculation of the stress intensity factor for the kinking crack.

Now consider the case when the crack tip is at rest at $x = 0$ and there are no loads acting on the body for $t < 0$. At time $t = 0$, the crack tip begins to move at a constant speed v_c. At the instant that the crack begins to extend, concentrated forces with magnitude n but opposite sign appear at the crack tip, one on each crack face as shown in Fig. 2. For $t > 0$ the crack tip moves in the positive x direction with speed v_c, while the concentrated forces move in the same direction with speed $u < v_c$, as shown in Fig. 2. The forces have equal magnitudes, each being the same general linear function of time (i.e. $mt+n$), m and n are arbitrary parameters which are independent of x and t. The boundary conditions on the plane $z = 0$ are

$$\tau_{zy}^{F}(x,0,t) = (mt+n)\Delta(x-ut)H(t) \qquad \text{for } -\infty < x < v_c t, \quad (2.6)$$

$$w^F(x,0,t) = 0 \qquad \text{for } v_c t < x < \infty, \quad (2.7)$$

where Δ is the Dirac delta function. The formulation is completed by specifying zero initial conditions and by requiring the solution satisfy the equation governing the motion of an elastic solid (1.1). Following Freund[24], the result is most easily obtained by considering the two cases $m = 1$, $n = 0$ and $m = 0$, $n = 1$ separately, and then adding the results for

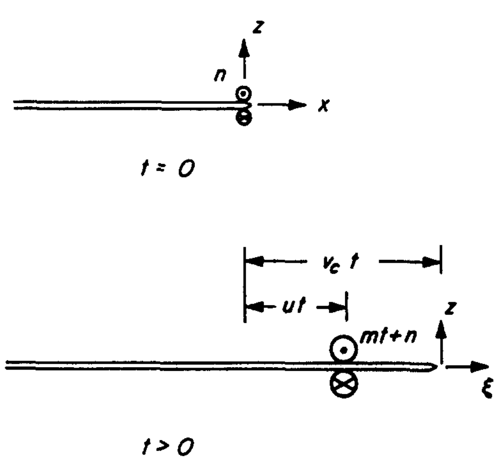

Fig. 2. The crack face loading corresponding to the fundamental solution.

each case. For $m = 0$, $n = 1$, the boundary conditions are

$$\tau_{zy}^{1}(x,0,t) = \Delta(x-ut)H(t) \qquad \text{for } -\infty < x < v_c t, \tag{2.8}$$

$$w^{1}(x,0,t) = 0 \qquad \text{for } v_c t < x < \infty, \tag{2.9}$$

and subject to zero initial conditions. It is found to be convenient to eliminate the coordinate x from the formulation in favor of a new coordinate ξ defined by

$$\xi = x - v_c t.$$

The (ξ, z)-coordinate system is then fixed with respect to the moving crack tip as shown on Fig. 2. The one-sided Laplace transform with respect to time and the two-sided Laplace transform with respect to x are defined by

$$\hat{w}^{1}(\xi,z,s) = \int_0^{\infty} w^{1}(\xi,z,t)\,\mathrm{e}^{-st}\,\mathrm{d}t, \tag{2.10}$$

$$\tilde{w}^{1}(\lambda,z,s) = \int_{-\infty}^{\infty} \hat{w}^{1}(\xi,z,s)\,\mathrm{e}^{-s\lambda\xi}\,\mathrm{d}\xi. \tag{2.11}$$

Applying the transform to the governing wave equation, the general solution for outgoing waves is

$$\tilde{w}^{1}(\lambda,z,s) = A(s,\lambda)\,\mathrm{e}^{-s\beta z}, \qquad \mathrm{Re}\,\beta \geqslant 0, \tag{2.12}$$

where

$$\begin{aligned} \beta(\lambda) &= (b^2-\lambda^2+b^2\lambda^2/d^2-2b^2\lambda/d)^{1/2} = \beta_+(\lambda)\beta_-(\lambda), \\ \beta_\pm(\lambda) &= [b\pm\lambda(1\mp b/d)]^{1/2}, \qquad d = 1/v_c. \end{aligned} \tag{2.13}$$

To ensure Re $\beta \geqslant 0$ everywhere in the λ-plane, branch cuts are introduced from $b_1 = b/(1+b/d)$ to ∞, and from $-b_2 = -b/(1-b/d)$ to $-\infty$. $A(s,\lambda)$ is an unknown function which can be evaluated from the transformed boundary conditions. The Laplace transform is now applied to the boundary conditions (2.8) and (2.9) yielding

$$\tilde{\tau}_{zy}^{1}(\lambda,0,s) = [h/s(h-\lambda)] + F_+(\lambda,s), \tag{2.14}$$

$$\tilde{w}^{1}(\lambda,0,s) = G_-(\lambda,s), \tag{2.15}$$

where h is the inverse of the relative speed between the moving load and the crack tip,

$$h = 1/(v_c-u).$$

The functions F_+ and G_-, which are as yet unknown functions of s and λ. For the transform F_+ to exist, F_+ must be regular in the half-plane Re $(\lambda) > -b_2$ and similarly, G_- must be regular in the half-plane Re $(\lambda) < b_1$. The function $A(s,\lambda)$ is eliminated from (2.14) and (2.15), yielding the single equation of the standard Wiener–Hopf type,

$$-\mu s\beta_- G_- - \frac{h}{s(h-\lambda)\beta_+(h)} = \frac{h}{s(h-\lambda)}\left[\frac{1}{\beta_+(\lambda)} - \frac{1}{\beta_+(h)}\right] + \frac{F_+}{\beta_+(\lambda)}. \tag{2.16}$$

Applying the well-known analytic continuation argument, the function $F_+(\lambda)$ and

$G_-(\lambda)$ can be completely determined and the transforms can be inverted. We find

$$G_-(\lambda, s) = A(s, \lambda) = \frac{-h}{\mu s^2 \beta_-(\lambda)(h-\lambda)\beta_+(h)}, \tag{2.17}$$

$$F_+(\lambda, s) = \frac{h}{s(\lambda-h)}\left[1 - \frac{\beta_+(\lambda)}{\beta_+(h)}\right]. \tag{2.18}$$

Then the shear stress on the plane $z = 0$, after formal inversion of the transforms, is

$$\tau^1_{zy}(\xi, 0, t) = -\frac{h\beta(-t/\xi)}{\xi\pi(h+t/\xi)\beta_-(-t/\xi)\beta_+(h)}, \qquad t > b_2\xi, \tag{2.19}$$

and

$$\lim_{\xi\to 0^+}\left[\xi^{1/2}\tau^1_{zy}(\xi, 0, t)\right] = -\frac{h(1-b/d)^{1/2}}{\pi[b+h(1-b/d)]^{1/2}t^{1/2}}. \tag{2.20}$$

In the same way, we can construct the solution for the case $m = 1$, $n = 0$,

$$\tau^2_{zy}(\xi, 0, t) = \int_{b_2}^{t/\xi} \frac{h^2\beta(-\lambda)}{\pi\beta_-(-\lambda)(\lambda+h)^2}\left[\frac{h+\lambda}{\beta_+(h)}\right]_h \mathrm{d}\lambda, \tag{2.21}$$

where the subscript h denotes differentiation with respect to h. The behavior of (2.21) as $\xi \to 0$ is obtained by applying a standard Tauberian theorem. The value of the limit is

$$\lim_{\xi\to 0^+}[\xi^{1/2}\tau^2_{zy}(\xi, 0, t)] = -\frac{h^2(1-b/d)^{3/2}t^{1/2}}{\pi[b+h(1-b/d)]^{3/2}}. \tag{2.22}$$

With these fundamental solutions at hand, it is possible to determine the first-order approximation for the stress intensity factor.

3. STRESS INTENSITY FACTOR OF KINKING CRACK

After the stress wave arrives at the original crack tip, the newly kinked crack begins to extend at a constant rate at some finite delay time t_f after diffraction has occurred. In order to extend, the crack must negate the tractions at $\theta = \delta$ and $0 \leqslant r \leqslant v_c(t-t_f)$ that are found from the diffraction and incident fields:

$$\tau_{\theta y}(r, \theta, t) = \tau^D_{\theta y} + \tau^I_{\theta y} \qquad \text{for } 0 \leqslant r \leqslant v_c(t-t_f), \tag{3.1}$$

$$\tau^I_{\theta y} = -\tau_0 \cos(\alpha - \theta), \tag{3.2}$$

where $\tau^I_{\theta y}$ represents the incident field and $\tau^D_{\theta y}$ is the diffraction field as shown in (2.3). The first-order approximation of the dynamic stress intensity factor for a kinked crack can be expressed by the stress intensity factor for a straight crack propagating in its own plane subjected the negative of the tractions given in (3.1) on the crack faces:

$$\tau_{zy} = 0 \qquad \text{for } \bar{x} < 0, \tag{3.3}$$

$$\tau_{zy} = -\tau_{\theta y}(\bar{x}/t, \theta = \delta) \qquad \text{for } 0 < \bar{x} < v_c(t-t_f). \tag{3.4}$$

It is observed that $\tau_{\theta y}(\bar{x}/t, \delta)$ depends only on the ratio $\bar{x}/t$ for fixed δ. Using the superposition described by Freund[24], this means that any fixed stress level radiates out along the $\bar{x}$-axis at a constant ratio $u = \bar{x}/t$ for $t \geqslant 0$, where u varies between zero and the shear wave speed. The resultant force due to stress levels with speed between u and $u+\mathrm{d}u$ is $t\tau_{\theta y}(\bar{x}/t, \delta)\,\mathrm{d}u$ and acts at $x = ut$. The time and the place that the force element $t\tau_{\theta y}(\bar{x}/t, \delta)\,\mathrm{d}u$ appears through the moving tip are $t^* = ht_f/d$ and $x^* = ut^*$. Hence the magnitude of this force element is

$$t^*\tau_{\theta y}(1/u, \delta)\,\mathrm{d}u + (t - t^*)\tau_{\theta y}(1/u, \delta)\,\mathrm{d}u.$$

The first term gives the magnitude of the force element at the instant it appears through the crack tip at t^*, and the second term indicates the magnitude increases linearly in time. The stress intensity factor for this kind of force element [i.e. (2.6)] is the summation of (2.20) and (2.22):

$$\frac{K^F(m,n,t)}{\sqrt{2\pi}} = -\frac{nh(1-b/d)^{1/2}}{\pi[b+h(1-b/d)]^{1/2}t^{1/2}} - \frac{mh^2(1-b/d)^{3/2}t^{1/2}}{\pi[b+h(1-b/d)]^{3/2}}. \tag{3.5}$$

Hence the first-order approximation of the stress intensity factor for the kinked crack due to the diffraction field can be obtained by integration over all possible values of u. We get

$$K^D(t, v_c, \delta) = \int_0^{v_c(t-t_f)/t} K^F(m = -1, n = -t^*, t - t^*)\tau_{\theta y}^D\left(\frac{1}{u}, \delta\right)\mathrm{d}u. \tag{3.6}$$

If we replace the variable of integration u in (3.6) by h, we find that the sum of these two terms is the derivative with respect to h of a single term. The superposition integral for the stress intensity factor then takes the form

$$K^D(t, v_c, \delta) = -\int_d^{d^*} 2\left[\frac{2t_f(1-b/d)}{\pi d}\right]^{1/2}\left\{\frac{(d^*-h)^{1/2}}{[b+h(1-b/d)]^{1/2}}\right\}_h \tau_{\theta y}^D\left(\frac{h}{v_c h - 1}, \delta\right)\mathrm{d}h, \tag{3.7}$$

where

$$d^* = t/(v_c t_f).$$

If we apply the integration by parts, neither the integrated term nor the remaining integral will exist, even though the sum exists. To get around this difficulty, by the method given in [24], the lower limit of integration in (3.7) is replaced by $d+\varepsilon$ as $\varepsilon \to 0$. The expression is then integrated by parts, and those terms which are singular at $\varepsilon = 0$ will exactly cancel each other. The desired result can be obtained by taking the limit as $\varepsilon \to 0$.

Integration by parts of (3.7) yields the sum of three terms

$$K^D(t, v_c, \delta) = \lim_{\varepsilon \to 0} 2B\left[\frac{2t_f(1-b/d)}{\pi d}\right]^{1/2}\left\{\frac{(d^*-d)^{1/2}}{\pi d^{1/2}}\left[\frac{2d}{b^{1/2}\varepsilon^{1/2}}\cos\frac{\theta}{2} - \frac{\pi}{2}(C_1 + C_2)\right]\right.$$
$$\left. + \int_{d+\varepsilon}^{d^*}\frac{(d^*-h)^{1/2}}{[b+h(1-b/d)]^{1/2}}(\tau_{\theta y}^D)_h\,\mathrm{d}h\right\} + o(1), \tag{3.8}$$

where

$$(\tau_{\theta y}^{D})_h = -\frac{[b+h(1-b/d)]^{1/2}[Ab(v_c h-1)+h\cos(\delta/2)]d^{3/2}}{\pi b^{1/2}\{h[1-(b/d)\sin\Theta_1]+b\sin\Theta_1\}\{h[1+(b/d)\sin\Theta_2]-b\sin\Theta_2\}(h-d)^{3/2}}, \tag{3.9}$$

$$\Theta_1 = \delta-\alpha, \qquad \Theta_2 = \delta+\alpha,$$

$$A = \sin\delta\sin(\delta/2)+\sin\alpha\cos(\delta/2), \qquad B = \frac{\tau_0\cos\alpha}{(1+\sin\alpha)^{1/2}},$$

$$C_1 = \frac{\sqrt{1-\sin\Theta_1}}{\cos\alpha}\left[\cos\left(\alpha-\frac{\delta}{2}\right)+\sin\frac{\delta}{2}\right], \qquad C_2 = \frac{\sqrt{1+\sin\Theta_1}}{\cos\alpha}\left[\cos\left(\alpha+\frac{\delta}{2}\right)-\sin\frac{\delta}{2}\right].$$

The second term in (3.8) comes from the last two terms in (2.4) which makes the significant contribution to the solution. This provides the real motivation for deriving the asymptotic expansion (2.4). Evaluation of the real integral in the third term of (3.8) becomes possible by converting it into a line integral in the complex h-plane and the closed form solution can be found by changing the contour of integration.

As shown in Fig. 3, the integrand has branch-point singularities at $h = d, d^*$ and two simple poles at

$$h_1 = -\frac{b\sin\Theta_1}{1-(b/d)\sin\Theta_1},$$

$$h_2 = \frac{b\sin\Theta_2}{1+(b/d)\sin\Theta_2}.$$

The integrand is single-valued in the plane cut along the real axis between d and d^*. Applying Cauchy's integral theorem, the value of the integral along Γ equals the residue at h_1 and h_2 minus the value of integral along Γ_ε minus the integral along a circle of infinitely large radius. Since the integrand is $O(h^{-2})$ as $|h| \to \infty$, the latter contribution is zero. We get

$$\int_{d+\varepsilon}^{d^*}\frac{(d^*-h)^{1/2}}{[b+h(1-b/d)]^{1/2}}(\tau_{\theta y}^{D})_h\,dh = \frac{[A+\sin\Theta_1\cos(\delta/2)][d^*(d-b\sin\Theta_1)+bd\sin\Theta_1]^{1/2}}{2(bd)^{1/2}\sin\delta\cos\alpha}$$

$$+\frac{[-A+\sin\Theta_2\cos(\delta/2)][d^*(d+b\sin\Theta_2)-bd\sin\Theta_2]^{1/2}}{2(bd)^{1/2}\sin\delta\cos\alpha}-\frac{2}{\pi}\cos\frac{\delta}{2}\left[\frac{(d^*-d)d}{b\varepsilon}\right]^{1/2}+o(1).$$

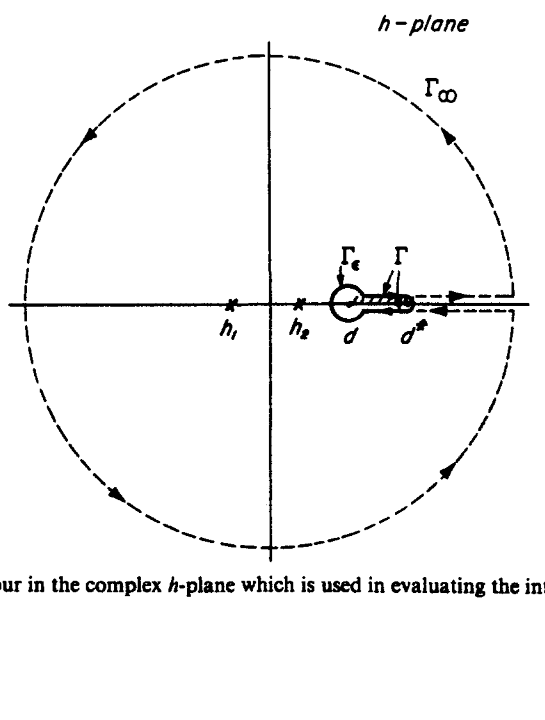

Fig. 3. The contour in the complex h-plane which is used in evaluating the intergral in (3.8).

The solutions in (3.8) are due to the diffraction field only. The stress intensity factor due to the incident field (3.2) can be easily obtained as follows

$$K^I(t, v_c, \delta) = \int_0^{v_c(t-t_f)} -K^F(m = 0, n = -1, t - t_f - x_0/v_c)\tau_{\theta_1}^I \, dx_0 \tag{3.10}$$

$$= 2\tau_0 \cos \Theta_1 \left[\frac{2(1-b/d)(t-t_f)}{\pi d}\right]^{1/2}.$$

Finally, the first-order approximation of the stress intensity factor for the kinking crack is the summation of the contribution due to diffraction and incident fields in (3.8) and (3.10). After working out the details, it is found that

$$K(t, v_c, \delta) = K^D(t, v_c, \delta) + K^I(t, v_c, \delta)$$

$$= 2\tau_0\left[\frac{2(1-b/d)}{\pi(1+\sin\alpha)}\right]^{1/2}\left\{\frac{1}{2b^{1/2}}\left(A_1\left[t-\frac{b}{d}(t-t_f)\sin\Theta_1\right]^{1/2}+A_2\left[t+\frac{b}{d}(t-t_f)\sin\Theta_2\right]^{1/2}\right)\right.$$

$$\left. -\frac{(t-t_f)^{1/2}}{2d^{1/2}}[A_1\sqrt{1-\sin\Theta_1}+A_2\sqrt{1+\sin\Theta_2}]+\left[\frac{(1+\sin\alpha)(t-t_f)}{d}\right]^{1/2}\cos\Theta_1\right\}, \tag{3.11}$$

where

$$A_1 = [\cos(\alpha-\delta/2)+\sin(\delta/2)], \qquad A_2 = [\cos(\alpha+\delta/2)-\sin(\delta/2)].$$

Relation (3.11) is the main result of this section. It is worth noting that when the kinking angle δ is zero, the new crack will propagate straight along the original crack path, and that the solution on (3.11) will be exact:

$$K(t, v_c, \delta = 0) = 2\tau_0 \cos\alpha\left[\frac{2(1-b/d)[t+(b/d)\sin\alpha(t-t_f)]}{\pi b(1+\sin\alpha)}\right]^{1/2}. \tag{3.12}$$

When the step function dependent stress wave loading strikes the original crack tip at time $t = 0$, the stress intensity factor of the stationary crack increases in proportion to $t^{1/2}$ as shown in (2.5). At a certain instant of delay time, the crack tip starts in motion. Assuming a critical energy balance criterion for crack initiation, the delay time t_f is related to the amount of energy γ needed to create a unit area of fracture surface quasistatically by

$$t_f = \frac{\mu\pi\gamma b(1+\sin\alpha)}{4\tau_0^2\cos^2\alpha}. \tag{3.13}$$

If we deal with the special case $t_f = 0$, that is the crack kinking at the instant that the incident stress wave strikes the original crack tip, the solution of (3.11) will give the same results as the numerical integrations in [25]. Figure 4 shows the dimensionless elastodynamic stress intensity factor for various values of the crack kinking angle δ and the normalized crack tip speed v_c/v_s, which is valid for $t_f = 0$ and incident stress angle $\alpha = \pi/4$. It is shown in [25] that the error for the first-order approximation compared to the exact values in [19] is less than 10% for any combination of angle of incident stress wave, kinked crack and propagation speed of the crack. The error increases as δ increases. However, for the kinking angle we are interested in, $-\pi/4 < \delta < \pi/4$, the error is even less than 2%. This good agreement with the exact solution, as mentioned is due to the fact that the contribution of the second-order approximation vanishes. It is unlikely that the error will be so small in the plane strain case.

In order to see the influence of introducing the finite delay time t_f to the dynamic stress intensity factor, we choose four kinking angles $\delta = \pm\pi/8$, $\pm\pi/4$ and plot K vs t_f/t in Figs.

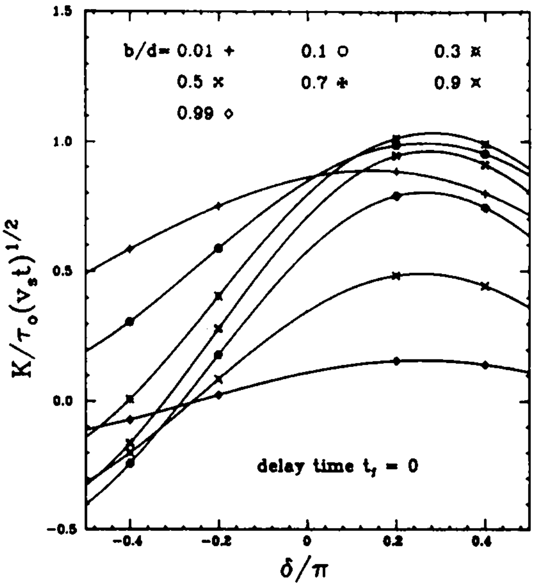

Fig. 4. The normalized stress intensity factor vs kinking angle for $t_f = 0$, $\alpha = \pi/4$ and different values of crack speeds.

5 and 6. The instant of initiation of the kink is at time $t = t_f$. When $t_f/t = 0$, the kinked crack has propagated for an infinite time compared to the delay time t_f. The results in [25] are valid for this special case. As we can see from these two figures, the dimensionless stress intensity factor is significantly different for $t_f/t = 0$ and $t_f/t = 1$. This difference increases as the kinked crack angle, the incident stress wave angle and the kinked crack speed increase. The difference is positive or negative depending on whether the kinked angle δ is positive or negative. If one wants to study the criterion for a crack kinking event, it is clear that the most significant time scale involved will be when crack kinking has just occurred, in other words, in the period of nondimensional time t_f/t near 1.

Besides the stress wave loading, step function loading that is suddenly applied uniformly

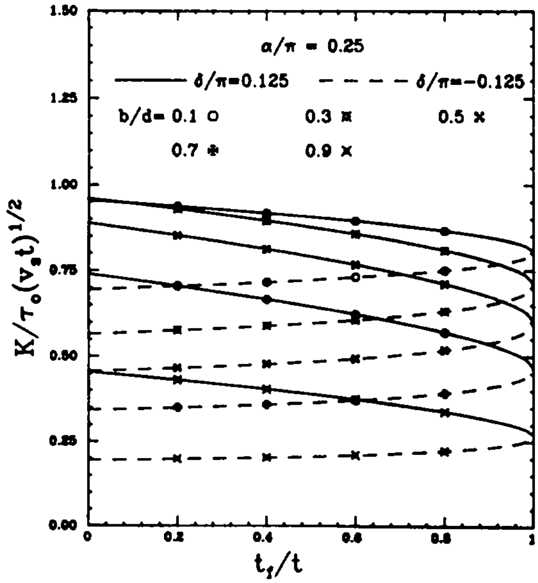

Fig. 5. The time history of stress intensity factor for $t > t_f$, $\alpha = \pi/4$, $\delta = \pm\pi/8$ and different values of crack speeds.

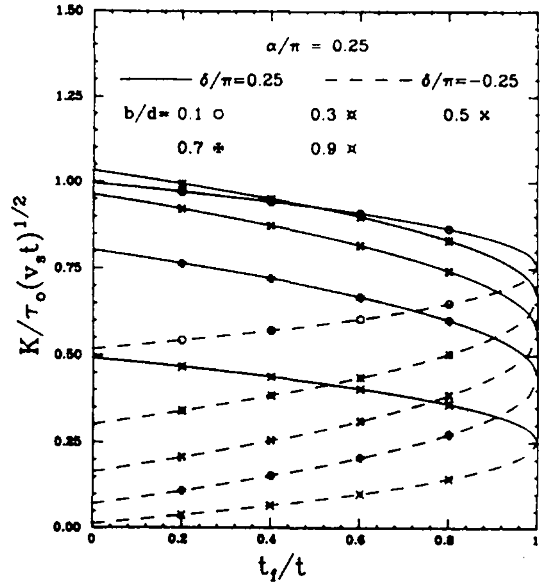

Fig. 6. The time history of stress intensity factor for $t > t_f$, $\alpha = \pi/4$, $\delta = \pm\pi/4$ and different values of crack speeds.

on the original crack faces is also of interest to us. For this case, we have

$$K^* = \tau_0 \cos\delta\left[\frac{2(1-b/d)}{\pi}\right]^{1/2}\left\{\left[\frac{t-(b/d)(t-t_f)\sin\delta}{b(1-\sin\delta)}\right]^{1/2} + \left[\frac{t+(b/d)(t-t_f)\sin\delta}{b(1+\sin\delta)}\right]^{1/2} - 2\left[\frac{t-t_f}{d}\right]^{1/2}\right\}. \tag{3.14}$$

Up to this point in the discussion, the time dependent of the stress wave loading profile is a simple step in time. In experiments, it is impossible to produce a true step profile and, instead, the stress wave loading pulse has a finite rise time. Suppose that at time $t = 0$ the stress wave loading has reached the original crack tip, and that the magnitude increases according to some function of time, say $f(t)$. After some finite rise time, say T, the magnitude of the loading is held constant for $t > T$. Suppose for the moment that the crack propagation delay time t_f is larger than the loading rise time T. In this case, the stress intensity factor of the kinking crack can be obtained for all time in a straightforward manner.

$$K = \int_0^{\tilde{t}} f'(s) K_S(t-s)\,\mathrm{d}s, \tag{3.15}$$

$$\tilde{t} = \begin{cases} t, & \text{for } t < T, \\ T, & \text{for } t > T, \end{cases}$$

where $K_S(t)$ is the stress intensity factor for unit step stress wave loading profile. If the rate of increase in magnitudes of the stress wave pulse is taken to be linear between $f(0) = 0$ and $f(T) = \tau_0$, then $f'(t) = \tau_0/T$ for $0 < t < T$ and $f'(t) = 0$ for $T < t$. The stress intensity factor of the whole time history is given by

$$K = \begin{cases} \dfrac{4\sqrt{2}\tau_0 \cos\alpha}{3\sqrt{\pi}[b(1+\sin\alpha)]^{1/2}T} t^{3/2} & \text{for } 0 < t < T, \\ \dfrac{4\sqrt{2}\tau_0 \cos\alpha}{3\sqrt{\pi}[b(1+\sin\alpha)]^{1/2}T}[t^{3/2} - (t-T)^{3/2}] & \text{for } T < t < t_f, \\ \dfrac{2\sqrt{2}\tau_0(1-b/d)^{1/2}}{\sqrt{\pi}(1+\sin\alpha)^{1/2}T}(K_1+K_2+K_3+K_4) & \text{for } t_f < t < \infty, \end{cases} \tag{3.16}$$

$$K_1 = \frac{A_1}{3b^{1/2}}\left\{\left[t-\frac{b}{d}(t-t_f)\sin\Theta_1\right]^{3/2}-\left[t-T-\frac{b}{d}(t-t_f)\sin\Theta_1\right]^{3/2}\right\},$$

$$K_2 = \frac{A_2}{3b^{1/2}}\left\{\left[t+\frac{b}{d}(t-t_f)\sin\Theta_2\right]^{3/2}-\left[t-T+\frac{b}{d}(t-t_f)\sin\Theta_2\right]^{3/2}\right\},$$

$$K_3 = -\frac{T(t-t_f)^{1/2}}{2d^{1/2}}[A_1\sqrt{1-\sin\Theta_1}+A_2\sqrt{1+\sin\Theta_2}],$$

$$K_4 = T\cos\Theta_1\left[\frac{(1+\sin\alpha)(t-t_f)}{d}\right]^{1/2},$$

where A_1, A_2 are given in (3.11).

It is emphasized at this point that the result shown above is valid only if the delay time t_f is greater than the rise time T. For the case when the crack kinked before the magnitude of the stress wave loading reaches its maximum level τ_0, that is for the case $t_f < T$, the solution is much more complicated to analyze and is discussed in [29].

The stress intensity factor history for the case $t_f > T$, given in (3.16), has been evaluated numerically and is shown in Figs. 7 and 8 for kinking crack tip speeds $v_c = 0.2v_s$ and $v_c = 0.5v_s$. Because the crack tip speed changes discontinuously at $t = t_f$, the stress intensity factor also changes abruptly at the same time.

4. ENERGY FLUXES INTO THE MOVING CRACK TIP

For Mode-III fracture, the energy fluxes into the propagating crack tip are related to the stress intensity factor by[9]

$$E = \frac{K^2}{2\mu d(1-b^2/d^2)^{1/2}} = \frac{\tau_0^2 t}{2\mu b^2}E^*. \tag{4.1}$$

For an incident step stress wave loading, E^* can be expressed as

$$E^* = \frac{2\hat{v}(1-\hat{v})^{1/2}}{\pi(1+\hat{v})^{1/2}(1+\sin\alpha)}\{A_1[1-\hat{v}(1-\hat{t})\sin\Theta_1]^{1/2}+A_2[1+\hat{v}(1-\hat{t})\sin\Theta_2]^{1/2}$$
$$-\sqrt{\hat{v}(1-\hat{t})}[A_1\sqrt{1-\sin\Theta_1}+A_2\sqrt{1+\sin\Theta_2}+2(1+\sin\alpha)^{1/2}\cos\Theta_1]\}^2, \tag{4.2}$$

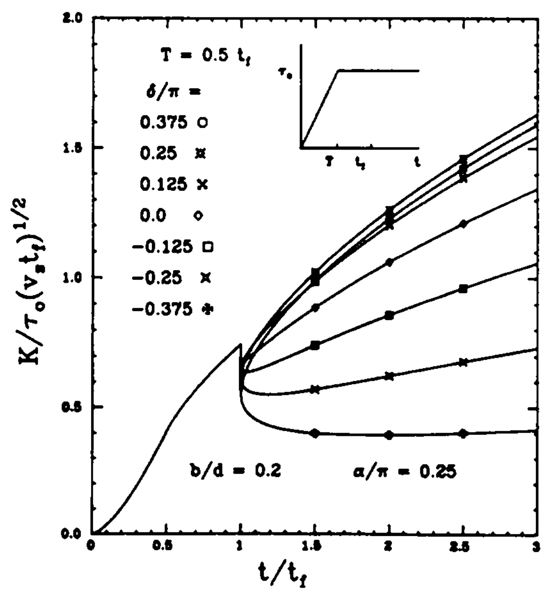

Fig. 7. The stress intensity factor history for incidence stress wave loading and the onset of crack growth at constant speed $v_c = 0.2v_s$ after some delay time t_f with various kinking angles.

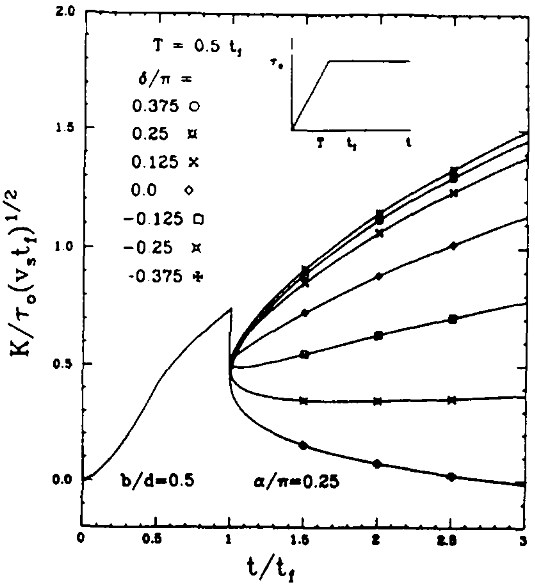

Fig. 8. The stress intensity factor history for incident stress wave loading and the onset of crack growth at constant speed $v_c = 0.5v_s$ after some delay time t_f with various kinking angles.

where

$$\hat{v} = b/d, \qquad \hat{t} = t_f/t,$$

and t_f can be obtained from (3.13). It is of interest to compute the kinking angle δ and normalized crack speed $\hat{v}$ at which E^* attains its maximum value for various values of α. The conditions for this to occur are

$$\partial E^*/\partial \hat{v} = 0, \qquad \partial^2 E^*/\partial \hat{v}^2 < 0, \tag{4.3}$$

and

$$\partial E^*/\partial \delta = 0, \qquad \partial^2 E^*/\partial \delta^2 < 0. \tag{4.4}$$

For the special case $t_f = 0$ and $\alpha = 0$ (or $\alpha = \pi/2$), the system of equations in (4.3) and (4.4) can be solved analytically

$$\delta = 0, \qquad \hat{v} = (\sqrt{5}-1)/2, \qquad E^*_{\max} = 0.765 \qquad \text{for } \alpha = 0,$$

and

$$\delta = \pi/2, \qquad \hat{v} = \sqrt{3/5}, \qquad E^*_{\max} = 0.545 \qquad \text{for } \alpha = \pi/2.$$

All other incident angles between 0 and $\pi/2$ can be solved numerically which is shown in Fig. 9. It compares very closely with the results in [25] for the five incident angles $\alpha = 0$, $\pi/8$, $\pi/4$, $3\pi/8$ and $\pi/2$. The general features are that the kinked crack speed $\hat{v}$ increases as α increases for E^* to achieve its maximum value $E^*_{\max}$, and for the whole range of α, the kinked angle δ is just slightly larger than α.

For the general case, the delay time is not zero. It is shown in Fig. 10 for two incident angles $\alpha = \pi/8$ and $\pi/4$ for the whole time history of a propagating crack tip such that $\hat{v}$ and δ satisfy eqns (4.3) and (4.4). It is worthy to note that for all cases[25] (i.e. $t_f = 0$) always overestimate $\hat{v}$ and $E^*_{\max}$. The kinking angle δ is approximately constant until t_f/t reaches 0.8, and thereafter it drops to zero rapidly.

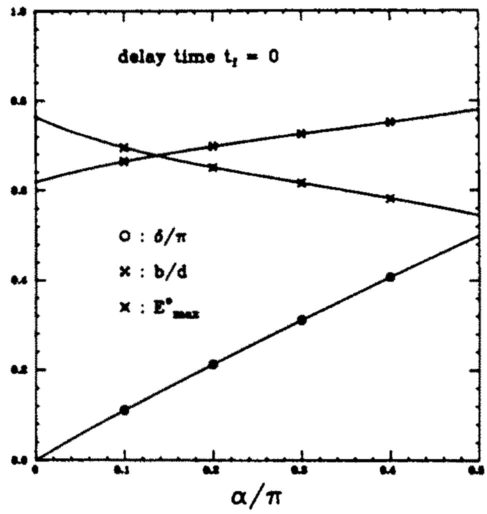

Fig. 9. Kinking angle and crack tip speed for E^*_{max}, for various angles of incident shear wave and $t_f = 0$.

DISCUSSION

Solutions of antiplane problems frequently suggest the proper steps for the approach to solve in-plane problems. There are some principal differences in the basic mechanisms of crack branching for the antiplane and inplane cases. Branches of a primary crack under pure Mode-I loading generally are subjected to both Mode-I and Mode-II loading conditions, whereas mixed loading conditions do not occur in antiplane strain. The extent of the inclusion of a delay time to the inplane case will be a great help to understand the criterion of the kinked crack from comparing with the available experimental result.

With the inclusion of a delay time, the solution of the fracture problem makes a great deal more sense physically and more closely models real material response. This allows something reasonably definite to be said about the initiation of crack growth. Frequently, an energy based fracture criterion is used to look at the initiation of crack tip motion. The criterion is based on the assumption that the energy release rate at initiation of fracture is a material parameter. For static fracture under small scale yielding conditions this is well established, but for dynamic fracture, it is not clear that this is a suitable criterion beyond the initiation phase. However, assuming that an energy based criterion is suitable, the

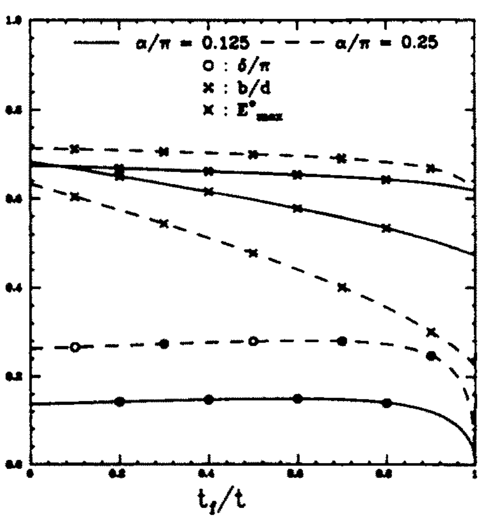

Fig. 10. Kinking angle and crack tip speed for E^*_{max}, for t_f not equal to zero and $\alpha = \pi/8, \pi/4$.

energy criterion suggests that the crack will choose to propagate in the direction and at velocity for which the energy flux into the crack tip has a maximum value. Based on this energy criterion and the results as shown in Fig. 10, it shows that for any angle of the incident shear wave, the crack will generally not kink. Achenbach and Kuo[30] also come to the same conclusion in their study on kinking of a crack in a prestressed body under the influence of incident stress waves. From experimental observations[1, 5–8], the crack does not kink at a distinct angle but curves gradually for some time before finding a fixed angle of propagation. The assumption of small scale yielding precludes us modeling this region in detail. However, the results in Fig. 10 for $t_f/t > 0$ seem to indicate that, if the details of how the crack kink are not too important, the angle of crack propagation may be at some angle δ not equal to zero. For dynamic crack propagation, the application of the maximum energy release rate criterion would require the elastodynamic stress intensity factors for arbitrary kinking angle and time varying crack tip speed. These are not available yet.

As observed by Ravi-Chandar and Knauss[1, 5–8] that the crack will most likely grow straight ahead of the original crack for crack tip speeds less than about 30% of the Rayleigh wave speed. The next step in the analysis to make it more physically correct, is to allow crack propagation before kinking or bifurcation. This problem is a great deal more difficult to solve but it must be attempted if we want to simulate the real physical response. The bifurcation event can be thought of as the rapidly propagating crack suddenly stopping and instantaneously (or after some delay time) bifurcating (or kinking). The stresses due to the sudden stopping must first be worked out for no kinking, then a similar method to the one used in this paper may again be applied. The problem mentioned will be an important analytical step in developing a criterion for kinking of a rapidly propagating crack.

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